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The Florida State University

Department
of
Statistics

Tallahassee, Florida
32306

ASYMPTOTIC BEHAVIOR OF INTERMEDIATE ORDER STATISTICS:
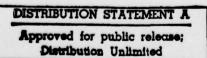THE INFINITE ENDPOINT CASE

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Vernon Watts

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ASYMPTOTIC BEHAVIOR OF INTERMEDIATE ORDER STATISTICS: THE INFINITE ENDPOINT CASE

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Vernon Watts

ABSTRACT

Suppose X_1, X_2, ... is a sequence of independent and identically distributed random variables with marginal distribution function $F(x) = P(X_1 \leq x)$ satisfying $F(x) > 0$ for all real x. Let $X_{k_n}^{(n)}$ denote the k_n^{th} smallest order statistic of the sample X_1, ..., X_n, where $k_n/n \to 0$ as $n \to \infty$. An almost sure representation of $X_{k_n}^{(n)}$ in terms of the empirical distribution function is established. The conditions imposed upon F include ... under which it is known that $X_{k_n}^{(n)}$ is asymptotically normal. From the representation the law of the iterated logarithm for $X_{k_n}^{(n)}$ is obtained. Examples illustrating the general result are presented.

1. <u>Introduction.</u> Suppose $\{X_n\}_{n \geq 1}$ is a sequence of independent and identically distributed (i.i.d.) random variables on (Ω, F, P) with marginal distribution function (d.f.) $F(x) = P(X_1 \leq x)$. Denote the kth smallest order statistic of the sample $X_1, \ldots, X_n$ by $X_k^{(n)}$. Let $\{k_n\}_{n \geq 1}$ be a sequence of integers satisfying $1 \leq k_n \leq n$ for each n and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$; $\{k_n\}$ is then called an intermediate rank sequence and $\{X_{k_n}^{(n)}\}_{n \geq 1}$ a sequence of intermediate order statistics.

In this paper we study the limiting properties of sequences $\{X_{k_n}^{(n)}\}_{n \geq 1}$. Our consideration is restricted to the case in which $F(x) > 0$ for all real x, that is, F has an infinite left endpoint. We first state a known result giving conditions under which $X_{k_n}^{(n)}$ is asymptotically normal. Then our primary interest is the development of an almost sure representation for $X_{k_n}^{(n)}$ in terms of the empirical d.f. of $X_1, \ldots, X_n$, which is analogous to our previous result obtained when $F(x_0) = 0$ for some $x_0 > -\infty$ (Watts (1977)). The procedure for deriving this essentially follows that used by Bahadur (1966) to establish the corresponding representation for sample λ-quantiles. Furthermore, we obtain the law of the iterated logarithm for $X_{k_n}^{(n)}$, and finally we illustrate the general representation with two examples.

2. <u>Asymptotic normality and the almost sure representation.</u> The basis for our investigation is the following theorem of Cheng (1965) giving sufficient conditions for the intermediate order statistic $X_{k_n}^{(n)}$ to be asymptotically normal. Here and throughout the paper we let x_n be defined by $F(x_n) = k_n/n$. The conditions upon the d.f. F insure that x_n is determined uniquely for all large n.

<u>Theorem 1.</u> Suppose for some real number c the two derivatives F' and F" exist with $F'(x) > 0$, for $x < c$. Also assume that for some $p \leq 1$ and some $M < \infty$,

(2.1) $\quad \dfrac{F(x_n)}{|x_n|^p \, F'(x_n)} < M$ and $\dfrac{F(x_n)\,|F''(x_n+y_n)|}{(F'(x_n))^2} < M$

for all large n, where $\{y_n\}$ is any sequence such that $y_n = o(|x_n|^p)$. Then

(2.2) $\quad P\left\{ \dfrac{n\,F'(x_n)}{k_n^{\frac{1}{2}}} \,(X_{k_n}^{(n)} - x_n) \le x \right\} \to \Phi(x)$

as $n \to \infty$, for all real x, where Φ is the standard normal d.f.

<u>Proof.</u> For any x, in order that (2.2) hold it is necessary and sufficient that

(2.3) $\quad \dfrac{n}{k_n^{\frac{1}{2}}}\,(F(u_n) - k_n/n) \to x$

as $n \to \infty$, where $u_n = u_n(x) = x_n + x\,k_n^{\frac{1}{2}}/n\,F'(x_n)$. (See Smirnov (1967).) Let $y_n = x\,k_n^{\frac{1}{2}}/n\,F'(x_n) = x\,F(x_n)/k_n^{\frac{1}{2}}\,F'(x_n)$. Since $p \le 1$ and $k_n \to \infty$ we have $y_n = o(x_n)$ by (2.1), so that $u_n \to -\infty$. Thus we may write

$$F(u_n) = F(x_n) + x\,k_n^{\frac{1}{2}}/n + \tfrac{1}{2}\,y_n^2\,F''(x_n + \theta_n\,y_n),$$

where $|\theta_n| \le 1$. Then since $y_n = o(|x_n|^p)$ and $F(x_n) = k_n/n$, the second condition in (2.1) leads to (2.3). $\square$

From now on we assume that F satisfies the conditions stated in Theorem 1. Moreover we impose the mild restriction that $k_n/\log^3 n \to \infty$. Let

$$a_n = C_o\,\dfrac{k_n^{\frac{1}{2}}\,\log^{\frac{1}{2}} n}{n\,F'(x_n)}$$

for some $C_o > 0$ and let $\{b_n\}$ be a sequence of positive integers satisfying

$$b_n \sim (k_n/\log n)^{\frac{1}{4}} \to \infty.$$

From (2.1) it follows that $a_n = o(x_n)$, so that $x_n + a_n \to -\infty$. For large n let $I_n = [x_n - a_n, x_n + a_n]$ and define

$$L_n = \sup\{F'(x): \; x \in I_n\}$$

and

$$M_n = \inf\{F'(x): \; x \in I_n\}.$$

Assume that

(2.4) $\qquad\qquad L_n/F'(x_n)$ and $F'(x_n)/M_n$ are bounded as $n \to \infty$.

Let $F_n(x, \omega) = n^{-1} \sum_{i=1}^{n} I(X_i \leq x)$ denote the empirical d.f. of $X_1, \ldots, X_n$. In the remainder of this section we establish the following result.

Theorem 2. Let $\{X_n\}$ be an i.i.d. sequence with marginal d.f. F satisfying the conditions of Theorem 1, and in addition, (2.4), where the intermediate rank sequence $\{k_n\}$ is such that $k_n/\log^3 n \to \infty$. Then

(2.5) $\qquad\qquad X_{k_n}^{(n)}(\omega) = x_n + \dfrac{k_n - n \, F_n(x_n, \omega)}{n \, F'(x_n)} + R_n(\omega),$

where $R_n(\omega) = 0\left(\dfrac{k_n^{\frac{1}{4}} \log^{3/4} n}{n \, F'(x_n)}\right)$ with probability one.

For large n let

$$G_n(x, \omega) = F_n(x, \omega) - F(x) - F_n(x_n, \omega) + k_n/n$$

and

$$H_n(\omega) = \sup\{|G_n(x, \omega)|: \; x \in I_n\}.$$

Most of the steps in the proof of Theorem 2 are contained in the following two lemmas.

Lemma 2.1. With probability one,

$$H_n(\omega) = 0(F'(x_n) \, a_n \, b_n^{-1}) = 0(n^{-1} \, k_n^{\frac{1}{4}} \, \log^{3/4} n)$$

as $n \to \infty$.

Proof: Let N_1 be such that x_n is uniquely determined for $n \geq N_1$. For $n \geq N_1$ and any integer r let $\eta_{r,n} = x_n + r \, a_n \, b_n^{-1}$. By the monotonicity of F and F_n we have for $x \in [\eta_{r,n}, \eta_{r+1,n}]$ that

$$F_n(\eta_{r,n}, \omega) - F(\eta_{r+1,n}) \leq F_n(x, \omega) - F(x) \leq F_n(\eta_{r+1,n}, \omega) - F(\eta_{r,n}).$$

Therefore

$$H_n(\omega) \leq \max\{|G_n(\eta_{r,n}, \omega)| : -b_n \leq r \leq b_n\}$$

$$+ \max\{F(\eta_{r+1,n}) - F(\eta_{r,n}) : -b_n \leq r \leq b_n - 1\}$$

(2.6)
$$= K_n(\omega) + \alpha_n,$$

say. We have $\eta_{r+1,n} - \eta_{r,n} = a_n \, b_n^{-1}$ and $\eta_{b_n,n} = x_n + a_n \to -\infty$. Then by (2.4),

(2.7)
$$\alpha_n = 0(F'(x_n) \, a_n \, b_n^{-1}).$$

Now for any $n \geq N_1$, and any r,

$$G_n(\eta_{r,n}) = \begin{cases} n^{-1} \sum_{i=1}^{n} I(x_n < X_i \leq \eta_{r,n}) - p_{r,n}, \ r \geq 0 \\ -n^{-1} \sum_{i=1}^{n} I(\eta_{r,n} < X_i \leq x_n) + p_{r,n}, \ r \leq -1, \end{cases}$$

where $p_{r,n} = |F(\eta_{r,n}) - k_n/n|$. Let $t_n = C_1 F'(x_n) \, a_n \, b_n^{-1}$ for some $C_1 > 0$. According to Bernstein's inequality (see Bahadur (1966)) we have

$$P(|G_n(\eta_{r,n})| \geq 2 \exp(-h_{r,n}),$$

where

$$h_{r,n} = \frac{n^2 t_n^2}{2\{n\, p_{r,n}(1-p_{r,n}) + \frac{nt_n}{3} \max\{p_{r,n},\ 1-p_{r,n}\}\}} .$$

We may choose $N_2 \geq N_1$ such that $n \geq N_2$ and $|r| \leq b_n$ imply $t_n \leq L_n a_n$ and $p_{r,n} \leq L_n a_n$. Thus for $n \geq N_2$ and $|r| \leq b_n$,

$$P(|G_n(\eta_{r,n})| \geq t_n) \leq 2 \exp(-\delta_n),$$

where

$$\delta_n = \frac{nt_n^2}{3 L_n a_n} = \frac{C_o C_1^2 F'(x_n)}{3 L_n} \log n(1 + o(1)).$$

If C_o and C_1 are chosen sufficiently large, then

$$P(|G_n(\eta_{r,n})| \geq t_n) \leq 2\, n^{-(1+\varepsilon)}$$

for some $\varepsilon > 0$ and for all $n \geq N_3$, say, so that

$$\sum_{n \geq N_1} P(K_n \geq t_n) < \infty.$$

Therefore $P(K_n \geq t_n$ infinitely often) $= 0$ by the Borel-Cantelli lemma, and since $t_n = C_1 F'(x_n) a_n b_n^{-1}$, the statement of the lemma follows from (2.6) and (2.7). □

Lemma 2.2. With probability one, $X_{k_n}^{(n)} \in I_n$ for all large n.

Proof: We first have

$$P(X_{k_n}^{(n)} < x_n - a_n) \leq P(\sum_{i=1}^n I(X_i \leq x_n + a_n) - np_n \geq k_n - np_n),$$

where $p_n = F(x_n - a_n)$. The right side does not exceed $2 \exp(-h)$, where

$$h = \frac{n^2(F(x_n) - F(x_n-a_n))^2}{2k_n} \geq \frac{n^2 M_n^2 a_n^2}{2k_n} \geq (1 + \epsilon) \log n$$

for some $\epsilon > 0$, if C_o is sufficiently large. Therefore

(2.8) $$\sum_{n \geq N_1} P(X_n < x_n - a_n) < \infty.$$

Similarly,

$$P(X_{k_n}^{(n)} > x_n + a_n) \leq P(\sum_{i=1}^{n} I(X_i > x_n + a_n) - np_n > n(1 - p_n) - k_n)$$

$$\leq 2 \exp(-h),$$

where now $p_n = 1 - F(x_n + a_n)$ and

$$h = \frac{n^2 M_n^2 a_n^2}{2n(1-p_n) + n L_n a_n}.$$

Since $F(x_n + a_n) \leq F(x_n) + L_n a_n$ and $\log n = o(k_n)$, and by (2.4), we have that

$$h \geq \frac{n^2 M_n^2 a_n^2}{5k_n} \geq (1 + \epsilon) \log n$$

for some $\epsilon > 0$ and for all large n. Thus

(2.9) $$\sum_{n \geq N_1} P(X_n > x_n + a_n) < \infty.$$

Combining (2.9) with (2.8) and applying the Borel-Cantelli lemma completes the proof. □

Proof of Theorem 2: From Lemmas 2.1 and 2.2 it follows that

$$F_n(X_{k_n}^{(n)}(\omega), \omega) - F(X_{k_n}^{(n)}(\omega)) - F_n(x_n, \omega) + k_n/n = O(F'(x_n) a_n b_n^{-1})$$

with probability one. Also, we have with probability one that for all large n,

$F_n(X_{k_n}^{(n)}(\omega), \omega) = k_n/n$ and

$$F(X_{k_n}^{(n)}(\omega)) = k_n/n + (X_{k_n}^{(n)}(\omega) - x_n) F'(x_n)$$

$$+ \tfrac{1}{2}(X_{k_n}^{(n)}(\omega) - x_n)^2 F''(x_n + a_n \theta_n(\omega)),$$

where $|\theta_n(\omega)| \le 1$. But $a_n b_n F''(x_n + a_n \theta_n(\omega))/F'(x_n) \to 0$, by (2.1). Therefore

$$(X_{k_n}^{(n)}(\omega) - x_n) F'(x_n) - k_n/n + F_n(x_n, \omega) = 0(F'(x_n) a_n b_n^{-1}),$$

and (2.5) follows. □

It may be noted that the above derivation of (2.5) requires that $\{k_n\}$ only satisfy $k_n/\log n \to \infty$. However if $k_n/\log^3 n \to \infty$, then (2.5) verifies the asymptotic normality of $X_{k_n}^{(n)}$, which follows from the usual central limit theory for row sums of triangular arrays of random variables, applied to $\{X_{n,i}\}$ defined by $X_{n,i} = (k_n/n - I(X_i \le x_n))/k_n^{\frac{1}{2}}$, $i = 1, \ldots, n$. Then it is clear that without the stronger restriction $k_n/\log^3 n \to \infty$ the rate of convergence of the remainder term R_n indicated in Theorem 2 may not be sufficiently rapid to insure that (2.5) is meaningful.

3. <u>The law of the iterated logarithm.</u> Kiefer (1972, Theorems 5 and 6) has established the law of the iterated logarithm for the intermediate order statistic $U_{k_n}^{(n)}$ from an i.i.d. sequence $\{U_n\}_{n \ge 1}$ uniformly distributed on the unit interval. Specifically, in his Theorem 5 he has shown that if $k_n \to \infty$ and $k_n/n \to 0$ monotonically, and if $k_n/\log \log n \to \infty$, then

$$P(\lim_{n \to \infty} \sup \pm \frac{T_n - k_n/n}{(2 k_n \log \log n)^{\frac{1}{2}}} = 1) = 1,$$

where $T_n = \sum_{i=1}^{n} I(U_i \le k_n/n)$. Suppose the marginal d.f. F of the sequence $\{X_n\}$ satisfies our conditions in the previous section, and moreover, is everywhere continuous, so that each $F(X_n)$ has the uniform distribution. Then using the almost sure relation

$$F_n(x_n, \omega) = n^{-1} \sum_{i=1}^{n} I(F(X_i) \le k_n/n),$$

valid for large n, along with the representation (2.5) we obtain the following result.

Theorem 3. Let $\{X_n\}$ be an i.i.d. sequence with continuous marginal d.f. F satisfying the conditions of Theorem 2. If $\{k_n\}$ satisfies $k_n \uparrow \infty$ and $k_n/n \downarrow 0$ as $n \to \infty$, along with the restriction $k_n(\log \log n)^2/\log^3 n \to \infty$, then

$$P(\limsup_{n \to \infty} \pm \frac{n \, F'(x_n)(X_{k_n}^{(n)} - x_n)}{(2 \, k_n \log \log n)^{\frac{1}{2}}} = 1) = 1.$$

4. Examples. We now illustrate the representation (2.5) with two examples. For each of these the conditions of Theorem 2 can be verified in a straightforward manner.

Example 1.
$$F(x) = \begin{cases} e^x, & x < 0 \\ 1, & x \ge 0. \end{cases}$$

Let $\{k_n\}$ satisfy $k_n/\log^3 n \to \infty$. For (2.1) we take p = 0. Then (2.5) holds, where

$$R_n(\omega) = O\left(\left(\frac{\log n}{k_n}\right)^{3/4}\right)$$

with probability one.

<u>Example 2.</u> $F(x) = \Phi(x)$.

We have the basic relation $F(x) \sim -x^{-1} F'(x)$ as $x \to -\infty$, from which it follows that

$$x_n \sim -(2 \log n/k_n)^{\frac{1}{2}},$$

for any $\{k_n\}$. Moreover, (2.1) is satisfied with $p = -1$. Then (2.5) holds, where

$$R_n(\omega) = O\left(\left[\frac{\log n}{k_n}\right]^{3/4} \left[\log \frac{n}{k_n}\right]^{-\frac{1}{2}}\right)$$

with probability one.

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20. ABSTRACT

Suppose $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables with marginal distribution function $F(x) = P(X_1 \leq x)$ satisfying $F(x) > 0$ for all real x. Let $X_{k_n}^{(n)}$ denote the k_n^{th} smallest order statistic of the sample $X_1, \ldots, X_n$, where $k_n/n \to 0$ as $n \to \infty$. An almost sure representation of $X_{k_n}^{(n)}$ in terms of the empirical distribution function is established. The conditions imposed upon F include those under which it is known that $X_{k_n}^{(n)}$ is asymptotically normal. From the representation the law of the iterated logarithm for $X_{k_n}^{(n)}$ is obtained. Examples illustrating the general result are presented.

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